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PRECISE ASYMPTOTIC FORMULAS FOR NONLINEAR EIGENVALUE PROBLEMS

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1. Introduction. We consider the following nonlinear two-parameter problem

$$\begin{aligned} &-u''(x)+\lambda u(x)^q=\mu u(x)^p, \ \ x\in I=(0,1),\\ &u(x)>0, \ \ x\in I, \qquad\qquad (1.1)\\ &u(0)=u(1)=0, \end{aligned}$$

where $1<q<p$ and $\lambda,\mu>0$ are parameters.

The purpose of this paper is to establish the asymptotic formulas for the eigencurve $\mu=\mu(\lambda)$ with the exact second term as $\lambda\to\infty$ by using a variational method. We also establish the critical relationship between p and q from a point of view of the decaying rate of the second term of $\mu(\lambda)$.

In Shibata [8], by using a standard variational framework (see Section 2), the variational eigencurve $\mu=\mu(\lambda)$ was defined to analyze $S_{\lambda,\mu}$ and the following asymptotic formula for $\mu(\lambda)$ as $\lambda\to\infty$ was established:

$$\mu(\lambda)=C_1\lambda^{(p+3)/(2p-q+3)}+o(\lambda^{(p+3)/(2p-q+3)}), \qquad (1.2)$$

$$C_1 = \left(\frac{(p+1)(q+3)}{(p+3)(q+1)} \frac{1}{\gamma^{p+1}} \frac{2}{p-q} \sqrt{\frac{\pi(q+1)}{2}} \left(\frac{p+1}{q+1}\right)^{\frac{q+3}{2(p-q)}} \frac{\Gamma\left(\dfrac{q+3}{2(p-q)}\right)}{\Gamma\left(\dfrac{p+3}{2(p-q)}\right)} \right)^{\frac{2(p-q)}{2p-q+3}},$$

$$\Gamma(r) = \int_0^\infty y^{r-1} e^{-y} dy \quad (r > 0). \tag{1.3}$$

By this formula, we understood the first term of $\mu(\lambda)$ as $\lambda \to \infty$. However, the remainder estimate of $\mu(\lambda)$ has not been obtained. The purpose here is to obtain *the exact second term* of $\mu(\lambda)$ as $\lambda \to \infty$. We emphasize that the second term depends deeply on the relationship between p and q, and the critical case is $p = (3q-1)/2$. As far as the author knows, this kind of criticality is new for two-parameter problems and great interest by itself. Finally, it should be mentioned that the asymptotic behavior of such eigencurve is also effected by the variational framework (cf. [6, 7]).

2. Main Results. Let $H_0^1(I)$ be the usual real Sobolev space. $\|u\|_r$ denotes the usual L^r-norm. For $u \in H_0^1(I)$

$$E_\lambda(u) := \frac{1}{2}\|u'\|_2^2 + \frac{1}{q+1}\lambda\|u\|_{q+1}^{q+1},$$

$$M_\gamma := \{u \in H_0^1(I) : \|u\|_{p+1} = \gamma\},$$

where $\gamma > 0$ is a *fixed constant.* For a given $\lambda > 0$, we call $\mu(\lambda)$ the variational eigenvalue when the following conditions (2.1)-(2.2) are satisfied:

$$(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}_+ \times \mathbf{R}_+ \times M_\gamma \ \ satisfies \ \ (1.1). \tag{2.1}$$

$$E_\lambda(u_\lambda) = \inf_{u \in M_\gamma} E_\lambda(u). \tag{2.2}$$

Then $\mu(\lambda)$ is obtained as a Lagrange multiplier and is represented explicitly as follows:

$$\mu(\lambda) = \frac{\|u'_\lambda\|_2^2 + \lambda\|u_\lambda\|_{q+1}^{q+1}}{\gamma^{p+1}}. \tag{2.3}$$

The existence of $\mu(\lambda)$ for a given $\lambda > 0$ is ensured in [8, Theorem 2.1] and $\mu(\lambda)$ is continuous for $\lambda > 0$ (cf. [8, Theorem 2.2]). Finally, let

$$K_1 := \left(\sqrt{2}\left(\frac{q+1}{p+1}\right)^{(q-1)/(2(p-q))} \frac{\Gamma\left(\frac{1}{q+1}\right)\Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi}(q+1)} C_1^{(q-1)/(2(p-q))}\right)^{2(q+1)/(q-1)},$$

$$K_2 := \frac{1}{2}\int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}}ds,$$

$$K_3 := \frac{2^{2(p+2)/(q+1)}}{q+1}\int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}}dy,$$

$$J_0 = \frac{\sqrt{\pi}}{p-q}\frac{q+3}{p+3}\frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)}.$$

Theorem 2.1. *(1) Assume $p > (3q-1)/2$. Then the following asymptotic formula holds as $\lambda \to \infty$:*

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)}\left\{1 + C_2(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))}\right\}, \tag{2.4}$$

where

$$C_2 = K_1\left(1 - \frac{2(p-q)K_2}{(2p-q+3)J_0}\right).$$

(2) Assume $p < (3q-1)/2$. Then as $\lambda \to \infty$:

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2p-q+3)}\left\{1 - C_3(1+o(1))\lambda^{-(p+1)/(q-1)}\right\}, \tag{2.5}$$

where

$$C_3 = \frac{2(p-q)}{(2p-q+3)J_0}K_3K_1^{(2p-q+3)/(2(q+1))}.$$

(3) Assume $p = (3q-1)/2$. Then as $\lambda \to \infty$:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 - C_4(1+o(1))\lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))} \log \lambda \right\}, \tag{2.6}$$

where

$$C_4 = \frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2 J_0} K_1.$$

The basic idea of the proof is as follows. Put

$$\begin{aligned} \nu(\lambda) &= \lambda^{\frac{p-1}{2(p-q)}} \mu(\lambda)^{\frac{1-q}{2(p-q)}}, \\ w_\lambda(t) &= \left(\frac{\mu(\lambda)}{\lambda}\right)^{\frac{1}{p-q}} u_\lambda(x), \quad t = \nu(\lambda)\left(x - \frac{1}{2}\right). \end{aligned} \tag{2.7}$$

Then it follows from (1.1) that w_λ satisfies

$$\begin{aligned} &-w_\lambda''(t) = w_\lambda(t)^p - w_\lambda(t)^q, \;\; t \in I_{\nu(\lambda)} := \left(-\frac{1}{2}\nu(\lambda), \frac{1}{2}\nu(\lambda)\right), \\ &w_\lambda(t) > 0, \;\; t \in I_{\nu(\lambda)}, \\ &w_\lambda\left(\pm\frac{1}{2}\nu(\lambda)\right) = 0. \end{aligned} \tag{2.8}$$

Then by [8, Lemma 5.1],

$$\nu(\lambda) \to \infty \tag{2.9}$$

as $\lambda \to \infty$. Put $z_\lambda = w_\lambda/\|w_\lambda\|_\infty$. Then it is easy to see from (2.3) that

$$\begin{aligned} \mu(\lambda) &= \frac{\lambda^{(p+3)/(2(p-q))}\mu(\lambda)^{-(q+3)/(2(p-q))}(\|w_\lambda'\|_2^2 + \|w_\lambda\|_{q+1}^{q+1})}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))}\mu(\lambda)^{-(q+3)/(2(p-q))}\|w_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}} \\ &= \frac{\lambda^{(p+3)/(2(p-q))}\mu(\lambda)^{-(q+3)/(2(p-q))}\|w_\lambda\|_\infty^{p+1}\|z_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}}. \end{aligned} \tag{2.10}$$

Therefore, it is crucial to study the asymptotic behavior of $\|w_\lambda\|_\infty$ and $\|z_\lambda\|_{p+1}$ as

3. Asymptotic behavior of $\|w_\lambda\|_\infty$. We put

$$\|w_\lambda\|_\infty = \left(\frac{p+1}{q+1}(1+\epsilon(\lambda))\right)^{1/(p-q)}. \tag{3.1}$$

Then by [8, (5.10), Lemma 5.2], we know that $\epsilon(\lambda) > 0$ and $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 3.1. *The following equality holds for $\lambda > 0$:*

$$\nu(\lambda) = \sqrt{2(q+1)}\left(\frac{p+1}{q+1}(1+\epsilon(\lambda))\right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)), \tag{3.2}$$

where

$$\begin{aligned} L(\epsilon) &= \int_0^1 \frac{1}{m(\epsilon, s)} ds, \\ m(\epsilon, s) &= \sqrt{s^{q+1} - s^{p+1} + \epsilon(1 - s^{p+1})} \quad (\epsilon > 0). \end{aligned} \tag{3.3}$$

Proof. Multiply the equation in (2.8) by w'_λ. Then for $t \in I_{\nu(\lambda)}$

$$\frac{d}{dt}\left(\frac{1}{2}(w'_\lambda(t))^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1}\right) = 0.$$

We know that $w_\lambda(0) = \|w_\lambda\|_\infty$ and $w'_\lambda(0) = 0$, since $u_\lambda(1/2) = \|u_\lambda\|_\infty$ and $u'_\lambda(1/2) = 0$. Then put $t = 0$ to obtain

$$\frac{1}{2}w'_\lambda(t)^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1} \equiv \frac{1}{p+1}\|w_\lambda\|_\infty^{p+1} - \frac{1}{q+1}\|w_\lambda\|_\infty^{q+1}.$$

Note that $w'_\lambda(t) < 0$ for $t \in (0, \nu(\lambda)/2)$, since $u'_\lambda(x) < 0$ for $x \in (1/2, 1)$. Then it follows from this and (3.1) that for $t \in (0, \nu(\lambda)/2)$

$$\begin{aligned} -z'_\lambda(t) &= \|w_\lambda\|_\infty^{(q-1)/2}\sqrt{\frac{2}{q+1}}\sqrt{z_\lambda(t)^{q+1} - z_\lambda(t)^{p+1} + \epsilon(\lambda)(1 - z_\lambda(t)^{p+1})} \\ &= \|w_\lambda\|_\infty^{(q-1)/2}\sqrt{\frac{2}{q+1}}m(\epsilon(\lambda), z_\lambda(t)). \end{aligned} \tag{3.4}$$

Put $s = z_\lambda$. Then (3.1) and (3.4) yield

$$\begin{aligned} \frac{\nu(\lambda)}{2} &= \int_0^{\nu(\lambda)/2} \frac{-z'_\lambda(t)}{\sqrt{\frac{2}{q+1}}\|w_\lambda\|_\infty^{(q-1)/2} m(\epsilon(\lambda), z_\lambda(t))} dt \\ &= \sqrt{\frac{q+1}{2}}\left(\frac{p+1}{q+1}(1+\epsilon(\lambda)\right)^{-(q-1)/(2(p-q))}\int_0^1 \frac{1}{m(\epsilon(\lambda), s)} ds. \end{aligned}$$

This implies (3.2). $\square$

Lemma 3.2. *For* $0 < \epsilon \ll 1$

$$L(\epsilon) = \frac{\Gamma\left(\frac{1}{q+1}\right)\Gamma\left(\frac{q-1}{2(q+1)}\right)}{(q+1)\sqrt{\pi}}\epsilon^{-(q-1)/(2(q+1))} + o(\epsilon^{-(q-1)/(2(q+1))}). \tag{3.5}$$

Proof. Put

$$L_1(\epsilon) := L(\epsilon) - \int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}}ds. \tag{3.6}$$

Put $s = \epsilon^{1/(q+1)}\tan^{2/(q+1)}\theta$. Then

$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}}ds \\
&= \frac{2}{q+1}\epsilon^{-(q-1)/(2(q+1))}\int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \sin^{-(q-1)/(q+1)}\theta\cos^{-2/(q+1)}\theta d\theta \\
&= \frac{2}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}\int_0^{\pi/2} \sin^{-(q-1)/(q+1)}\theta\cos^{-2/(q+1)}\theta d\theta \\
&= \frac{1}{q+1}(1+o(1))\epsilon^{-(q-1)/(2(q+1))}\frac{\Gamma\left(\frac{1}{q+1}\right)\Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi}}.
\end{aligned} \tag{3.7}$$

Next, we calculate $L_1(\epsilon)$. Note that for $0 \le s \le 1$

$$m(\epsilon, s) = \sqrt{s^{q+1}(1-s^{p-q}) + \epsilon(1-s^{p+1})} \ge \sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}. \tag{3.8}$$

By this, we obtain

$$\begin{aligned}
&|L_1(\epsilon)| \\
&= \int_0^1 \frac{(1+\epsilon)s^{p+1}}{m(\epsilon,s)\sqrt{s^{q+1}+\epsilon}(m(\epsilon,s)+\sqrt{s^{q+1}+\epsilon})}ds \\
&\le \int_0^1 \frac{(1+\epsilon)s^{p+1}}{\sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}\sqrt{s^{q+1}+\epsilon}(\sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}+\sqrt{s^{q+1}+\epsilon})}ds \\
&\le (1+\epsilon)\int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}(1+\sqrt{1-s^{p-q}})}ds \\
&\le 2\int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds \\
&= 2\int_0^\delta \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds + 2\int_\delta^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2}\sqrt{1-s^{p-q}}}ds \\
&:= I + II,
\end{aligned}$$

where $0 < \delta \ll 1$ is a fixed constant. Let $C_{j,\delta} > 0$ $(j = 1, 2, \cdots)$ be constants depending only on δ. Put $s = \sin^{2/(p-q)} \theta$. Then

$$\begin{aligned} II &\le \frac{2}{\delta^{3(q+1)/2}} \int_\delta^1 \frac{1}{\sqrt{1-s^{p-q}}} ds \\ &= \frac{2}{\delta^{3(q+1)/2}} \frac{2}{p-q} \int_{\sin^{-1}\delta^{(p-q)/2}}^1 \sin^{(2+q-p)/(p-q)} \theta d\theta \\ &\le C_{1,\delta}. \end{aligned} \tag{3.10}$$

Moreover, put $s = \epsilon^{1/(q+1)} t$. Then for $0 < \epsilon \ll 1$

$$\begin{aligned} I &\le \frac{2}{\sqrt{1-\delta^{p-q}}} \int_0^\delta \frac{\epsilon^{(p+1)/(q+1)} t^{p+1}}{\epsilon^{3/2}(t^{q+1}+1)^{3/2}} \epsilon^{1/(q+1)} dt \\ &\le 2 \frac{\delta^{p+1}}{\sqrt{1-\delta^{p-q}}} \epsilon^{(2p-3q+1)/(2(q+1))} = o(\epsilon^{-(q-1)/(2(q+1))}). \end{aligned} \tag{3.11}$$

By (3.9)–(3.11), we have

$$|L_1(\epsilon)| = o(\epsilon^{-(q-1)/(2(q+1))}).$$

By this, (3.6) and (3.7), we obtain (3.5). □

Lemma 3.3. *As* $\lambda \to \infty$

$$\epsilon(\lambda) = K_1(1 + o(1))\lambda^{-2(p+1)(q+1)/((q-1)(2p-q+3))}. \tag{3.12}$$

Proof. By (1.2) and (2.7), we have

$$\begin{aligned} \nu(\lambda) &= \lambda^{(p-1)/(2(p-q))} \mu(\lambda)^{(1-q)/(2(p-q))} \\ &= C_1^{(1-q)/(2(p-q))}(1 + o(1))\lambda^{(p+1)/(2p-q+3)}. \end{aligned} \tag{3.13}$$

On the other hand, by Lemmas 3.1–3.2 and Taylor expansion, we have

$$\begin{aligned} \nu(\lambda) &= \sqrt{2(q+1)} \left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))} (1 + \epsilon(\lambda))^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)) \\ &= \sqrt{2} \left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))} \frac{\Gamma\left(\frac{1}{q+1}\right)\Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi}(q+1)} \epsilon(\lambda)^{-(q-1)/(2(q+1))}(1 + o(1)). \end{aligned}$$

By this and (3.13), we obtain (3.12). □

4. Asymptotic behavior of $\|z_\lambda\|_{p+1}$. By (3.4) and putting $s = z_\lambda(t)$, we have

$$\begin{aligned}
\|z_\lambda\|_{p+1}^{p+1} &= 2\int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1}dt \\
&= 2\int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1} \frac{-z_\lambda'(t)}{\|w_\lambda\|_\infty^{(q-1)/2}\sqrt{\frac{2}{q+1}m(\epsilon(\lambda), z_\lambda(t))}}dt \\
&= \frac{\sqrt{2(q+1)}}{\|w_\lambda\|_\infty^{(q-1)/2}} J(\epsilon(\lambda)),
\end{aligned} \tag{4.1}$$

where

$$J(\epsilon) := \int_0^1 \frac{s^{p+1}}{m(\epsilon, s)}ds \qquad (\epsilon > 0). \tag{4.2}$$

Therefore, we study the precise asymptotics of $J(\epsilon)$ as $\epsilon \to 0$. Put $s = \sin^{2/(p-q)}\theta$. Then as $\epsilon \to 0$

$$\begin{aligned}
J(\epsilon) \to J(0) &= \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{1-s^{p-q}}}ds \\
&= \frac{2}{p-q}\int_0^{\pi/2} \sin^{(p+3)/(p-q)}\theta d\theta \\
&= \frac{\sqrt{\pi}}{p-q}\frac{q+3}{p+3}\frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)} \\
&= J_0.
\end{aligned} \tag{4.3}$$

We use here the formulas

$$\begin{aligned}
&\int_0^{\pi/2} \sin^r\theta d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)} \quad (r > -1), \\
&\Gamma(r+1) = r\Gamma(r).
\end{aligned} \tag{4.4}$$

Therefore, put

$$\begin{aligned}
J_1(\epsilon) &:= J(\epsilon) - J_0 := -\epsilon J_2(\epsilon), \\
J_2(\epsilon) &:= \int_0^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds.
\end{aligned} \tag{4.5}$$

We study the asymptotic behavior of $J_2(\epsilon)$ as $\epsilon \to 0$.

Lemma 4.1. *(1) If* $p > (3q-1)/2$, *then* $J_2(\epsilon) \to K_2$ *as* $\epsilon \to 0$.

(2) If $p < (3q-1)/2$, *then as* $\epsilon \to 0$

$$J_2(\epsilon) = K_3(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))}. \tag{4.6}$$

(3) If $p = (3q-1)/2$, *then as* $\epsilon \to 0$

$$J_2(\epsilon) = -\frac{1}{2(q+1)}(1+o(1))\log\epsilon. \tag{4.7}$$

Proof. (1) Since $p > (3q-1)/2$, we have $(2p-3q-1)/2 > -1$. Therefore, by Lebesgue's convergence theorem, as $\epsilon \to 0$

$$J_2(\epsilon) \to \frac{1}{2}\int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}}ds = K_2.$$

This completes the proof.

(2) *Step 1.* Assume that $p < (3q-1)/2$. We introduce $J_3(\epsilon)$ to approximate $J_2(\epsilon)$:

$$\begin{aligned} J_3(\epsilon) &:= \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon})}ds \\ &= J_4(\epsilon,\delta) + J_5(\epsilon,\delta) \\ &:= \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon})}ds \\ &\quad + \int_\delta^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon})}ds, \end{aligned} \tag{4.8}$$

where $0 < \delta \ll 1$ is a fixed small constant. We study the asymptotic behavior of J_3, J_4 and J_5 as $\epsilon \to 0$. Note that $0 < (2p-2q+2)/(q+1) < 1$, since $p < (3q-1)/2$. Then put $s = \epsilon^{1/(q+1)}\tan^{2/(q+1)}\theta$ and $y = \tan(\theta/2)$ to obtain

$$\begin{aligned} J_3(\epsilon) &= \frac{2}{q+1}\epsilon^{(2p-3q+1)/(2(q+1))}\int_0^{\tan^{-1}1/\sqrt{\epsilon}} \frac{\tan^{(2p-2q+2)/(q+1)}\theta}{1+\sin\theta}d\theta \\ &= K_3(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))}. \end{aligned} \tag{4.9}$$

Similarly, we obtain

$$J_4(\epsilon,\delta) = K_3(1+o(1))\epsilon^{(2p-3q+1)/(2(q+1))}, \quad J_5(\epsilon,\delta) \le \frac{1}{\delta^{q+1}}. \tag{4.10}$$

Since $p < (3q-1)/2$, this along with (4.9) implies that $J_3(\epsilon)/J_4(\epsilon,\delta) \to 1$ as $\epsilon \to 0$ for a fixed δ.

Step 2. We show that as $\epsilon \to 0$

$$\frac{J_2(\epsilon)}{J_3(\epsilon)} \to 1. \tag{4.11}$$

Let an arbitrary $0 < \delta \ll 1$ be fixed. Put

$$\begin{aligned} J_2(\epsilon) &= J_6(\epsilon,\delta) + J_7(\epsilon,\delta) \\ &:= \int_0^\delta \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds \\ &\quad + \int_\delta^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds. \end{aligned} \tag{4.12}$$

Then for $0 < \epsilon \ll 1$

$$|J_7(\epsilon,\delta)| \le C_{2,\delta} \int_\delta^1 \frac{1-s^{p+1}}{(1-s^{p-q})^{3/2}}ds \le C_{3,\delta}. \tag{4.13}$$

Moreover, by (3.8), we obtain

$$\begin{aligned} (1-\delta^{p+1}) \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon})}ds \le J_6(\epsilon,\delta) \\ \le \frac{1}{(1-\delta^{p-q})^{3/2}} \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}+\epsilon}(s^{(q+1)/2}+\sqrt{s^{q+1}+\epsilon})}ds. \end{aligned}$$

This implies

$$(1-\delta^{p+1})J_4(\epsilon,\delta) \le J_6(\epsilon,\delta) \le \frac{1}{(1-\delta^{p-q})^{3/2}}J_4(\epsilon,\delta). \tag{4.14}$$

By (4.10), (4.13) and (4.14), we see that $J_7(\epsilon,\delta) = o(J_6(\epsilon,\delta))$ as $\epsilon \to 0$ for a fixed δ, since $p < (3q-1)/2$. Then by (4.9), (4.10) and (4.12)–(4.14),

$$\begin{aligned} (1-\delta^{p+1}) \le \liminf_{\epsilon\to 0} \frac{J_6(\epsilon,\delta)}{J_4(\epsilon,\delta)} = \liminf_{\epsilon\to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \le \limsup_{\epsilon\to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \\ = \limsup_{\epsilon\to 0} \frac{J_6(\epsilon,\delta)}{J_4(\epsilon,\delta)} \le \frac{1}{(1-\delta^{p-q})^{3/2}}. \end{aligned} \tag{4.15}$$

By letting $\delta \to 0$, we obtain (4.11). Then by (4.9) and (4.11), we obtain (4.6).

(3) If $p = (3q-1)/2$, then by the asymptotic formula

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^3}\right) \quad (x \gg 1),$$

and Taylor expansion of $\tan x$ at $x = \pi/4$ and (4.9), we obtain (4.7) by direct calculation. $\square$

5. Proof of Theorem 2.1. By (2.10), (3.1), (4.1) and (4.5), we have

$$\begin{aligned} \mu(\lambda)^{(2p-q+3)/(2(p-q))} &= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}} \lambda^{(p+3)/(2(p-q))} \|w_\lambda\|_\infty^{(2p-q+3)/2} J(\epsilon(\lambda)) \\ &= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}} \lambda^{(p+3)/(2(p-q))} \left(\frac{p+1}{q+1}\right)^{(2p-q+3)/(2(p-q))} \\ &\quad \times (1+\epsilon(\lambda))^{(2p-q+3)/(2(p-q))} (J_0 - \epsilon(\lambda) J_2(\epsilon(\lambda))). \end{aligned} \tag{5.1}$$

Moreover, it is easy to check that

$$\left(\frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\right)^{2(p-q)/(2p-q+3)} \frac{p+1}{q+1} J_0^{2(p-q)/(2p-q+3)} = C_1.$$

By this, (5.1) and Taylor expansion, we obtain

$$\begin{aligned} \mu(\lambda) &= C_1 \lambda^{(p+3)/(2p-q+3)} \\ &\quad \times \left(1 + \epsilon(\lambda) - \frac{2(p-q)}{(2p-q+3)J_0}(1+o(1))\epsilon(\lambda) J_2(\epsilon(\lambda))\right). \end{aligned} \tag{5.2}$$

Then by Lemma 3.3, Lemma 4.1 and direct calculation, we obtain Theorem 2.1. Thus the proof is complete. $\square$

References

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